University of California, Riverside
Department of Mathematics

Final Exam

Mathematics 8B - First Year of Calculus
Sample 3

Instructions: This exam has a total of 200 points. You have 3 hours. You must show all your work to receive full credit You may use any result done in class. The points attached to each problem are indicated beside the problem. You are not allowed books, notes, or calculators. Answers should be written as $\sqrt{2}$ as opposed to 1.4142135....

The following formulae may be useful: for any A, B, we have

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B),\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B),\\
\tan(A+B) &= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}, \qquad \tan\left(\frac{A}{2}\right)=\frac{\sin(A)}{1+\cos(A)}=\frac{1-\cos(A)}{\sin(A)},\\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1-\cos(A)}{2}}, \qquad \cos\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1+\cos(A)}{2}},\\
\sin(2A) &= 2\sin(A)\cos(A), \qquad \cos(2A)=\cos^2(A)-\sin^2(A),\\
\cos(2A) &= 2\cos^2(A)-1, \qquad \cos(2A)=1-2\sin^2(A),\\
\tan(2A) &= \frac{2A}{1-\tan^2(A)}
\end{aligned}$$

1. (a) (10 points) Find the exact value of the composition $\tan(\cos^{-1}(\frac{1}{2}))$.

   (b) (10 points) The angle α in quadrant IV satisfies

   $$\cot(\alpha)=-\frac{1}{3}$$

   Find the values of $\sin(\alpha)$, $\cos(\alpha)$, $\tan(\alpha)$, $\csc(\alpha)$ and $\sec(\alpha)$.

2. Evaluate the following limits.

(a) (5 points) $\lim_{x\to 1} \frac{x-1}{\sqrt{3x+1}-2}$

(b) (5 points) $\lim_{x\to 3} \frac{x^2-6x+9}{x^2-9}$

(c) (5 points) $\lim_{x\to 0} \frac{\sin(4x)}{3x\cos(5x)}$

(d) (5 points) $\lim_{x\to \infty} \frac{2+3x^2}{5x^2-4x}$

3. Find the derivatives of the following functions.

   (a) (7 points) $f(x) = \frac{x^2-1}{x^2+1}$

   (b) (6 points) $g(x) = 3\sin(4x) + 4\cos(3x)$

   (c) (7 points) $h(x) = \tan(\sqrt{1+x^3})$

4. Show that the equation $5x - 2 + \sin(x) = 0$ has exactly one real root with the following two steps.

   (a) (10 points) Use the Intermediate Value Theorem to show that the equation has a root between 0 and π.

   (b) (10 points) Assuming the equation has at least two real roots, derive a contradiction using the Rolle's Theorem.

5. (20 points) Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$3x^2 + xy + y^2 = 5 \text{ at } (1, -2)$$

6. (20 points) A kite 30 (meters) above the ground moves horizontally at a speed of 6 (m/s). At what rate is the length of string increasing when 50 (meters) of the string has been let out?

7. (20 points) Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = \frac{x-1}{x^2-2x+2} \text{ on the interval } [-1, 3]$$

8. Given the function $f(x) = x^3 - 6x^2 + 5$,

(a) (6 points) Find the intervals of increase and decrease.

(b) (6 points) Find the local maximum and local minimum values.

(c) (6 points) Find the intervals of concave upward and concave downward.

(d) (6 points) Find the inflection point(s).

(e) (6 points) Use the the above information (1) to (4) to sketch the graph of $y = f(x)$.

9. (20 points) Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 5 cm and 12 cm if two sides of the rectangle lie along the legs.

10. (10 points) Find the most general antiderivative of the function

$$f(x) = 2\sqrt{x} + \cos(2x),$$

and check your answer by differentiation.